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H^∞ Collocated Control of Structural Systems: An Analytical Bound Approach

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The H^∞ norm analysis and output feedback control synthesis problems for structural systems with collocated sensors and actuators are examined. By using a particular solution of the bounded real lemma for an open-loop collocated structural system, we obtain an explicit expression to compute an upper bound on the H^∞ norm of such systems. Then, for the corresponding output feedback H^∞ control synthesis problem we obtain an explicit parameterization of the output feedback control gains that achieve a desired H^∞ norm bound. These results have obvious computational advantages for large-scale systems where standard H^∞ analysis and control design methods are computationally intractable. Computational examples demonstrate the advantages of the proposed results.

I. Introduction

THE control of structural systems with collocated sensors and actuators has been shown to provide great advantages from a stability, passivity, robustness, and an implementation viewpoint. For example, collocated control can easily be achieved in a space structure when an attitude rate sensor is placed at the same point as a torque actuator.[1,2] Collocation of sensors and actuators leads to symmetric transfer functions. Several other classes of engineering systems, such as circuit systems, chemical reactors, and power networks, can be modeled as systems with symmetric transfer functions. Stabilization, robustness, model reduction, and control of such systems have been examined recently.[3–6]

State-space H^∞ control based on the standard Riccati equation approach or the recent linear-matrix-inequality (LMI) formulation is now a well-developed control synthesis tool. The optimal static-state feedback and full-order dynamic output feedback H^∞ control synthesis problems can be solved using iterations on the corresponding Riccati solutions or via the computational solution of a convex LMI optimization problem.[7–9] On the other hand, the static output feedback and the fixed-order dynamic output feedback H^∞ control synthesis problems are difficult computational problems because they require the solution of (nonconvex) bilinear matrix inequalities or LMIs with coupling rank constraints.[10–12]

In this work, we examine the H^∞ control analysis and the symmetric output feedback H^∞ control synthesis problems for systems with symmetric transfer functions. The objective of the paper is to show that, by exploiting the particular structure of these systems, explicit bounds for the H^∞ control problems can be obtained. To this end, a particular solution of the bounded real lemma is proposed, and an explicit expression for an upper bound on the H^∞ norm of such a symmetric system that requires only the computation of the maximum eigenvalue of a symmetric matrix is obtained. Subsequently, we derive an explicit parameterization for the output feedback H^∞ control gains that guarantee this bound. The proofs of the results are purely algebraic based on simple matrix algebra tools. This work generalizes the results of Ref. 13, which consider systems with state-space symmetry, which is a special case of transfer function symmetry. However, in Ref. 13 the corresponding algebraic results are exact, although in the present paper the results provide a conservative bound on the H^∞ norm.

The notation to be used in this paper is as follows: given a real matrix N, the orthogonal complement $N^\perp$ is defined as the (possibly nonunique) matrix with maximum row rank that satisfies $N^\perp N = 0$ and $N^\perp N^{\perp T} > 0$. Hence, $N^\perp$ can be computed from the singular value decomposition of N as follows: $N^\perp = T U_2^T$, where T is an arbitrary nonsingular matrix and U_2 is defined from the singular value decomposition of N

$$N = [U_1 \quad U_2]\begin{bmatrix}\Sigma_1 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}V_1^T\\ V_2^T\end{bmatrix}$$

The standard notation $>$ ($<$) is used to denote the positive (negative) definite ordering of symmetric matrices. The ith eigenvalue of a real symmetric matrix N will be denoted by $\lambda_i(N)$, where the ordering of the eigenvalues is defined as $\lambda_{\max}(N) = \lambda_1(N) \geq \lambda_2(N) \geq \cdots \geq \lambda_n(N)$. The maximum singular value of a (not necessarily square) matrix N will be denoted by $\sigma_{\max}(N)$, which is also its spectral norm $\|N\|$. N^+ will denote the Moore–Penrose generalized inverse of a matrix N.

II. Collocated H^∞ Control Analysis Problem

Consider the following vector second-order representation of a structural system with collocated sensors and actuators

$$M\ddot{q} + D\dot{q} + Kq = Fu, \qquad y = F^T\dot{q} \tag{1}$$

where $q(t) \in \mathbb{R}^n$ is the generalized coordinate vector, $u(t) \in \mathbb{R}^m$ is the input vector, and $y(t) \in \mathbb{R}^k$ is the measured output vector. The matrices M, D, and K are symmetric positive-definite matrices that represent the structural system mass, damping, and stiffness distribution, respectively. For simplicity, we will assume that the input distribution matrix F for full column rank. The system has a state-space realization as follows:

$$\dot{x} = Ax + Bu, \qquad y = Cx \tag{2}$$

with

$$A = \begin{bmatrix}0 & I\\ -M^{-1}K & -M^{-1}D\end{bmatrix}, \qquad B = \begin{bmatrix}0\\ M^{-1}F\end{bmatrix}$$

$$C = [0 \quad F^T] \tag{3}$$

where $x = [q^T \quad \dot{q}^T]^T$. Notice that the transfer function $H(s)$ of system (2) and (3)

$$H(s) = sF^T(Ms^2 + Ds + K)^{-1}F$$

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is symmetric, that is, $H(s) = H^T(s)$. System (2) and (3) is an externally symmetric state-space realization, that is, there exists a nonsingular matrix T such that

$$A^T T = TA, \qquad C^T = TB \tag{4}$$

This class of systems is more general than the class of internally or state-space symmetric systems that satisfy the symmetry conditions (4) with a positive-definite transformation matrix T (Ref. 6). Obviously, state-space symmetry implies external symmetry, but the converse is not true, that is, there exist symmetric transfer matrices for which there is no internally symmetric realization. An explicit analytical solution of the H^∞ control problem for internally symmetric systems has been presented in Ref. 13.

Recall that the H^∞ norm of the system (2) is given by

$$\|H\|_\infty = \sup_{\omega \in \mathbb{R}} \sigma_{\max}\{H(j\omega)\}$$

where $H(s) = C(sI - A)^{-1}B$ is the transfer function of the system and $\sigma_{\max}$ denotes the maximum singular value of a matrix. It is well known that for a stable LTI system, its H^∞ norm can be approximated iteratively, for example, using a bisection method.[14] The next result shows that for a vector second-order realization (1), an upper bound on its H^∞ norm can be computed using a simple explicit formula.

Theorem 1: Consider the vector second-order system realization (1). The system has an H^∞ norm γ that satisfies

$$\gamma < \bar{\gamma} = \lambda_{\max}(F^T D^{-1} F) \tag{5}$$

To prove this result, recall the following bounded real lemma (BRL) characterization of the H^∞ norm of a system.[15]

Lemma 2: A stable system (2) has an H^∞ norm less than or equal to γ if and only if there exists a matrix $P > 0$ satisfying

$$\begin{bmatrix} A^T P + PA & PB & C^T \\ B^T P & -\gamma I & 0 \\ C & 0 & -\gamma I \end{bmatrix} \le 0 \tag{6}$$

Now, Theorem 1 follows from the symmetric system BRL condition and the following algebraic result.[16]

Lemma 3: Consider matrices Γ and Q such that Γ has full column rank and Q is symmetric positive definite. Then $Q \ge \Gamma\Gamma^T$ if and only if

$$\lambda_{\max}(\Gamma^T Q^{-1} \Gamma) \le 1$$

Proof of Theorem 1: The result follows from the bounded real Lemma 2 by using the following Lyapunov matrix:

$$P = \begin{bmatrix} K & 0 \\ 0 & M \end{bmatrix} \tag{7}$$

Using matrix (7), BRL 2 provides

$$\begin{bmatrix} -2D & F & F \\ F^T & -\gamma I & 0 \\ F^T & 0 & -\gamma I \end{bmatrix} \le 0$$

Application of the Schur complement formula[17] results in the following condition:

$$-2D - [F \quad F] \begin{bmatrix} -\gamma I & 0 \\ 0 & -\gamma I \end{bmatrix}^{-1} \begin{bmatrix} F^T \\ F^T \end{bmatrix} = -D + \frac{1}{\gamma} F F^T \le 0$$

that is,

$$D \ge (1/\gamma) F F^T$$

Now, application of Lemma 3 provides the bound (5). ∎

III. H^∞ Control Synthesis Problem

Consider the following controlled vector second-order system:

$$M\ddot{q} + D\dot{q} + Kq = F(u + w), \qquad z = F^T \dot{q}, \qquad y = F^T \dot{q} \tag{8}$$

where $q(t) \in \mathbb{R}^n$ is the generalized coordinate vector, $u(t) \in \mathbb{R}^m$ is the control input vector, $w(t) \in \mathbb{R}^m$ is the external and disturbance input, $y(t) \in \mathbb{R}^k$ is the measured output vector, and $z(t) \in \mathbb{R}^k$ is the performance output vector. The collocated H^∞ control synthesis problem is to design a symmetric static feedback gain $G = G^T$ such that the output feedback control law

$$u = -Gy \tag{9}$$

renders the closed-loop system stable with an H^∞ norm less than a given scalar $\gamma > 0$.

The closed-loop system of the plant (8) and the controller (9) is

$$M\ddot{q} + (D + FGF^T)\dot{q} + Kq = Fw \tag{10}$$

$$z = F^T \dot{q} \tag{11}$$

The following result provides an explicit expression for the output feedback gains that guarantee a closed-loop H^∞ norm less than a given bound γ.

Theorem 4: Consider the vector second-order system (8). For any $\gamma > 0$ there exists a symmetric output feedback control law (9) to provide a closed-loop H^∞ norm less than γ.

1) If F is square and invertible, then G can be selected as

$$G \ge (1/\gamma)I - F^{-1} D F^{-1T} \tag{12}$$

2) If FF^T is singular, then G can be selected as

$$G \ge F^+[DF^{\perp T}(F^\perp D F^{\perp T})^{-1} F^\perp D - D + (1/\gamma) FF^T] F^{+T} \tag{13}$$

The preceding result indicates that the control gain should be positive semidefinite if γ is small and the system damping is low, but it does not have to be positive semidefinite if γ is large and the damping is high.

Proof of Theorem 4: Applying BRL 2 to the closed loop system (10) and (11) by using the Lyapunov matrix (7) results in

$$\begin{bmatrix} -2(D + FGF^T) & F & F \\ F^T & -\gamma I & 0 \\ F^T & 0 & -\gamma I \end{bmatrix} \le 0$$

Then, by using the Schur complement formula[17] we obtain the following condition for the control gain G:

$$FGF^T + D - (1/\gamma) FF^T \ge 0 \tag{14}$$

If F is square and invertible, then condition (12) follows. If FF^T is singular, we follow an approach similar to the one in the proof of the generalized Finsler's lemma.[12] By using a congruent transformation with the matrix

$$T = \begin{bmatrix} F^+ \\ F^\perp \end{bmatrix}$$

condition (14) results in

$$\begin{bmatrix} G + F^+ D F^{+T} - 1/\gamma & F^+ D F^{\perp T} \\ F^\perp D F^{+T} & F^\perp D F^{\perp T} \end{bmatrix} \ge 0$$

where we have used that $F^+ F = I$ and $F^\perp F = 0$. Noting that $F^\perp D F^{\perp T} > 0$, the condition for positive semidefiniteness of the preceding matrix provides[17]

$$(G + F^+ D F^{+T} - 1/\gamma) - (F^+ D F^{\perp T})(F^\perp D F^{\perp T})^{-1}(F^\perp D F^{+T}) \ge 0$$

which results in the bound (13). ∎

IV. Examples

A. Single-Degree-of-Freedom Case

To demonstrate and motivate the preceding results, consider the one-degree-of-freedom (1-DOF) case ($n = 1$), where $q(t)$, $u(t)$, and $y(t)$ are scalar quantities in Eq. (1). For this scalar case the magnitude of the frequency response function (FRF) of the system (1) is

$$|H(j\omega)| = \frac{F^2|\omega|}{\sqrt{(K - M\omega^2)^2 + D^2\omega^2}}$$

where M, D, and K represent the scalar mass, damping, and stiffness coefficients of the system. Notice that the preceding FRF magnitude satisfies the following bound at all frequencies:

$$|H(j\omega)| \leq F^2|\omega|/D|\omega| = F^2/D$$

that is, $\|H\|_\infty \leq F^2/D$. This bound is precisely the one provided in Theorem 1 for this system. In fact, in this scalar case the preceding bound provides the exact H^∞ norm of the system, that is, $\|H\|_\infty = F^2/D$.

For the controlled vector second-order scalar case described by Eq. (8) with a scalar velocity feedback (9), the magnitude of the closed-loop FRF $H_{\text{cl}}(j\omega)$ is given by

$$|H_{\text{cl}}(j\omega)| = \frac{F^2|\omega|}{\sqrt{(K - M\omega^2)^2 + (D + GF^2)^2\omega^2}}$$

Hence, at all frequencies

$$|H_{\text{cl}}(j\omega)| \leq \frac{F^2|\omega|}{(D + GF^2)^2|\omega|} = \frac{F^2}{D + GF^2}$$

that is, $\|H_{\text{cl}}\|_\infty \leq F^2/(D + GF^2)$. Therefore, a closed-loop H^∞ norm bound $\|H_{\text{cl}}\|_\infty \leq \gamma$ is obtained if $F^2/(D + GF^2) \leq \gamma$, or

$$G \geq 1/\gamma - D/F^2$$

which is precisely the result of theorem 4. In this case γ provides the exact H^∞ norm of the closed-loop system, that is, $\|H_{\text{cl}}\|_\infty = \gamma$.

B. Three-DOF Spring-Mass-Damper System

Consider the structural system shown in Fig. 1 that consists of three masses interconnected with springs and dampers with the structural parameters $m_i = 1$, $d_i = \delta$, and $k_i = 1$ for $i = 1, 2, 3$.

The corresponding structural matrices of the system are as follows:

$$M = I_{3\times 3}, \qquad K = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 2\delta & -\delta & 0 \\ -\delta & 2\delta & -\delta \\ 0 & -\delta & \delta \end{bmatrix}$$

We seek to examine the values of the H^∞ norm bound (5) compared to the exact H^∞ norm value when the damping parameter δ of the system varies. The following figure (Fig. 2) shows this comparison, where the solid line represents the exact H^∞ norm of the system and the dotted line represents the H^∞ norm bound (5).

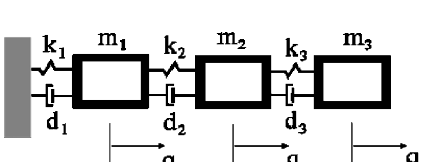

Fig. 1 Spring-mass-damper system.

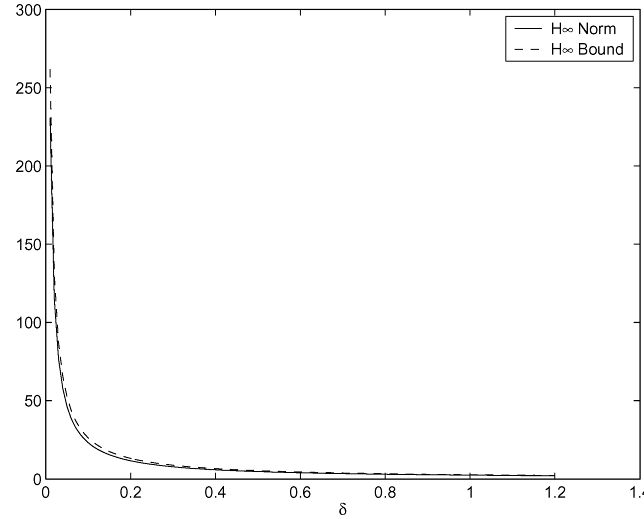

Fig. 2 Comparison of the H^∞ norm and the proposed H^∞ bound.

Now consider a control design problem for the same structural system as just discussed with $d_i = 1$. We assume an input matrix

$$F = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

The open-loop system has an H^∞ norm equal to 2.3681. We seek to find a symmetric output feedback gain matrix G such that the H^∞ norm of the closed-loop system is less than $\gamma = 0.5$. Theorem 4 provides a parameterization of such gains as follows:

$$G \geq \bar{G} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

Notice that $\bar{G}$ results in a closed-loop H^∞ norm equal to $0.4918 < 0.5$. For a desired $\gamma = 0.2$, Theorem 4 results in

$$G \geq \bar{G} = \begin{bmatrix} 3 & 1 \\ 1 & 4 \end{bmatrix}$$

and $\bar{G}$ results in a closed-loop H^∞ norm equal to $0.1988 < 0.2$.

C. International Space Station Collocated Control

The real benefit of the proposed bounds is evident in the analysis and control of very large-scale symmetric systems, such as large-scale structures and power networks, where standard H^∞ analysis and design tools are computationally prohibitive. To demonstrate this point, consider the finite element structural model for the assembly phase 8A-OBS of the International Space Station with collocated control and Rayleigh damping.[18] This model is in the form (1) with 360 DOF, that is, the corresponding state-space model (2) and (3) has 720 states. Computation of an H^∞ control design via standard Riccati equation or LMI methods is computationally intractable. In fact, it takes 3282.8 s to calculate the exact H^∞ norm of the system that equals 82.747. However, the proposed bound (5) provides an open-loop H^∞ norm bound of the system equal to $\bar{\gamma} = 83.102$, which takes only 0.501 s to calculate. In addition, a symmetric static output feedback gain to reduce this bound to, say, $\bar{\gamma} = 5$ is easily computed using the results of Theorem 4. It takes only 0.671 s to compute this control gain. The exact H^∞ norm of the closed-loop system is 4.9988, and it takes 1762.22 s to compute it using standard methods. The result of Theorem 1 provides a closed-loop H^∞ norm bound $\bar{\gamma} = 5$ in 0.17 s. The preceding computations have been performed in a 1.33 G Hz Athlon PC, and the corresponding results and computational times for different values of the desired closed-loop H^∞ norm bound $\bar{\gamma} = 5$, 1, 0.5, and 0.1 of the system are shown in Table 1. Figure 3 shows the open-loop maximum singular value

Table 1 Results for different values of the desired H^∞ norm bound $\bar{\gamma}$

Desired closed-loop H^∞ norm bound $\bar{\gamma}$	Time to calculate the feedback gain with Theorem 4, s	Exact H^∞ norm of the closed-loop system	Time to calculate the exact H^∞ norm, s	Time to calculate the H^∞ norm bound (5), s
5	0.671	4.9988	1762.220	0.1700
1	0.681	0.99997	1963.553	0.2099
0.5	0.671	0.499999	1961.891	0.1800
0.1	0.661	0.099999999	1926.120	0.1910

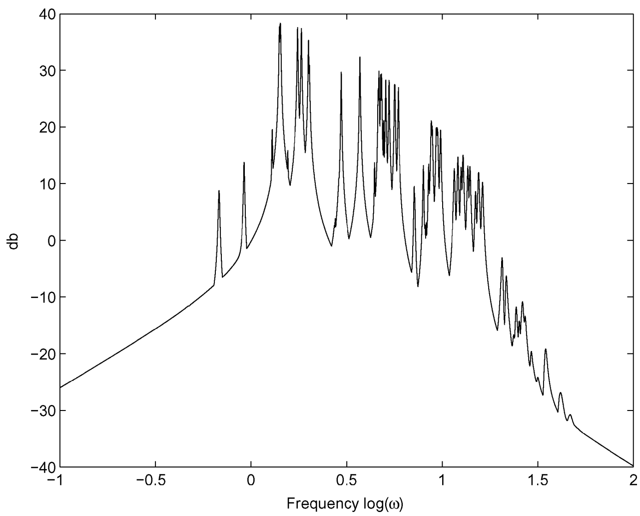

Fig. 3 Maximum singular value plot of the open-loop system.

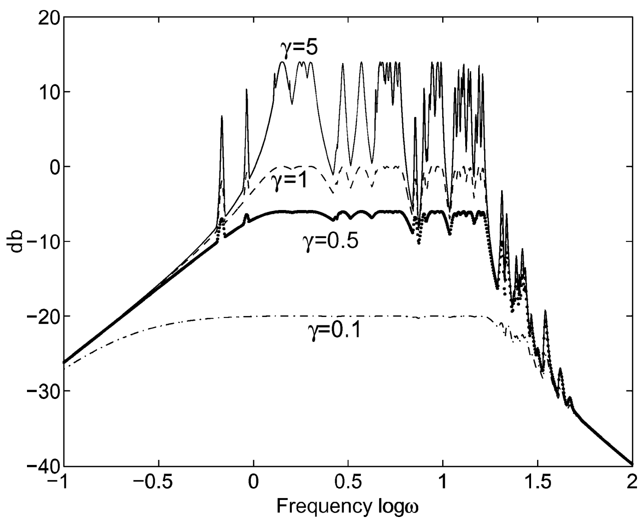

Fig. 4 Maximum singular value of the closed-loop system for different H^∞ norm bounds $\bar{\gamma}$.

(sigma) plot of the preceding system, and Fig. 4 shows the corresponding closed-loop singular value plots for $\bar{\gamma} = 5$, 1, 0.5, and 0.1 using the feedback gain formula in Theorem 4. It can be easily observed from these figures that the closed-loop system satisfies the given bounds.

V. Conclusions

We have obtained a simple explicit expression for an H^∞ norm bound of structural systems with collocated sensors and actuators. In addition, an explicit parameterization of symmetric output feedback gains that lead to a desired closed-loop H^∞ norm bound has been derived. The results provide easily computable guidelines for H^∞ analysis and control of collocated structural systems and are particularly useful for very large-scale systems where standard H^∞ analysis and design methods are compuationally intractable. The results are applicable to any system with a symmetric transfer function.

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